Увод в теорията на операторни алгебри с приложение в математиката и физиката
Introduction to Theory of Operator Algebras with Applications to Mathematics and Physics
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Lecture 6

Von Neumann Algebras (Алгебри на фон-Нойман)

1. Commutants.

Let $\mathcal{H}$ be a Hilbert space and $\mathcal{B}(\mathcal{H})$ be the C^*-algebra of all bounded operators on $\mathcal{H}$. For a subset $\mathcal{S} \subseteq \mathcal{B}(\mathcal{H})$:

$$\mathcal{S}' := \{A \in \mathcal{B}(\mathcal{H}) \mid [A,B] \equiv A\cdot B - B\cdot A = 0 \ (\forall B \in \mathcal{S})\}.$$

Then $\mathcal{S}'$ is a unital Banach subalgebra of $\mathcal{B}(\mathcal{H})$ (why?).
If $\mathcal{S}$ is $*$-invariant, i.e. $A \in \mathcal{S} \Rightarrow A^* \in \mathcal{S}$ then $\mathcal{S}'$ is a C^*-algebra (why?).
Additional properties: $\mathcal{S}_1 \subseteq \mathcal{S}_2 \Rightarrow \mathcal{S}_2' \subseteq \mathcal{S}_1'$; $\mathcal{S} \subseteq \mathcal{S}''$;
$\mathcal{S} \subseteq \mathcal{S}'' = \mathcal{S}^{(iv)} = \dots$; $\mathcal{S}' = \mathcal{S}^{(iii)} = \dots$

If $\mathcal{S}$ is $*$-invariant and $\mathcal{C}$ is the smallest C^*-algebra containing $\mathcal{S}$ then $\mathcal{S}' = \mathcal{C}'$: indeed $\mathcal{C} = \overline{\mathbb{C}[\mathcal{S}]}$.

Def. Von Newman algebra $\mathfrak{M} \subseteq \mathcal{B}(\mathcal{H})$ is a subalgebra of a form $\mathfrak{M} = \mathcal{S}'$ for a $*$-invariant subset $\mathcal{S} \subseteq \mathcal{B}(\mathcal{H})$. Then: $\mathfrak{M} = \mathfrak{M}''$.

The centre of the algebra is: $\mathcal{Z}(\mathfrak{M}) := \mathfrak{M} \cap \mathfrak{M}'$ - it is a commutative von Neumann algebra.
The von Neumann algebra is called factor iff $\mathcal{Z}(\mathfrak{M}) = \mathbb{C}.\mathbb{1}$.
A C^*-subalgebra $\mathcal{A} \subseteq \mathcal{B}(\mathcal{H})$ is called nondegenerate if $\overline{\mathcal{A}\mathcal{H}} = \mathcal{H}$.
in particular, if $\mathcal{A}$ is unital then $\mathcal{C}$ is nondegenerate.

Let $\mathcal{A}$ be a C^*-subalgebra of $\mathcal{B}(\mathcal{H})$, P be the orthogonal projection on $\overline{\mathcal{A}\mathcal{H}}$ then $PA = A \ \forall A \in \mathcal{A}$. $\Rightarrow (A^*)^* P = (A^*)^* \ (\forall A \in \mathcal{A})$, i.e.
$PA = AP = A$, $(1-P)A = A(1-P) = 0 \ (\forall A \in \mathcal{A})$. Thus: $A(\overline{\mathcal{A}\mathcal{H}}) \subseteq \overline{\mathcal{A}\mathcal{H}}$,
$A|_{\mathcal{A}\mathcal{H}^\perp} = 0 \ (\forall A \in \mathcal{A})$ and $\mathcal{A}|_{\overline{\mathcal{A}\mathcal{H}}} = \{A|_{\overline{\mathcal{A}\mathcal{H}}} \mid A \in \mathcal{A}\} \subseteq \mathcal{B}(\overline{\mathcal{A}\mathcal{H}})$
is nondegenerate and $\mathcal{A} = \mathcal{A}|_{\overline{\mathcal{A}\mathcal{H}}} \oplus 0$.

Also: $(\mathcal{A}\mathcal{H})^\perp = \{u \in \mathcal{H} \mid Au = 0 \ \forall A \in \mathcal{A}\}$.

Lemma 6.1 Let $\mathcal{S} \subseteq \mathcal{B}(\mathcal{H})$ be a $*$-invariant subset ($A \in \mathcal{S} \Rightarrow A^* \in \mathcal{S}$), $V \subseteq \mathcal{H}$ be a closed vector subspace and P be the orthogonal projection on V. Then V is $\mathcal{S}$-invariant (i.e., $\forall A \in \mathcal{S}: A(V) \subseteq V$) iff ($\Leftrightarrow$) $P \in \mathcal{S}'$.

Hint. Show first that it is enough to prove the lemma for $\mathcal{S} = \{A, A^*\}$, where $A \in \mathcal{B}(\mathcal{H})$ is arbitrary. Then the part "$\Leftarrow$" is trivial. The part "$\Rightarrow$" follows from $AP = PAP$ $\forall A \in \mathcal{S}$, which implies in turn $PA = PAP$ ($\forall A \in \mathcal{S}$) – why? $\square$

2. Reminder from topology: convergence and closure.

A comment for motivation:

In $\mathbb{R}$ the closure $\overline{S}$ of a set $S \subseteq \mathbb{R}$ can be obtained by taking all limits of convergent sequences lying in S. For a general Hausdorff topological space this is no longer true since the structure of the system of neighbourhoods of a point, the so called local bases, can be much more complicated. In general, such local bases have the charcteristic feature of being directed sets (a notion we shall remind below). For this reason it turns out that the necessary and sufficient generalization of the notion of convergent sequences, which is needed to produce the closures is the notion of a convergent net that is a "sequence" labeled by a directed set.

Def. A directed set (насочено множество) is a poset $(M, \prec)$ s.t. $\forall k = 1, 2, \dots$ $\forall \mu_1, \dots, \mu_k \in M$ $\exists \nu \in M$: $\mu_1 \preceq \nu, \dots, \mu_k \preceq \nu$.

A net (направление) in a topological space X is a family $\{x_\mu\}_{\mu \in M} \subseteq X$ labeled with a directed set M.

The net is called convergent iff $\exists x \in X$ s.t. for $\forall$ neighbourhood $\mathcal{U}$ of x $\exists \mu \in M$ s.t. $x_\nu \in \mathcal{U}$ for $\forall \nu \succeq \mu$.

In a Hausdorff space the limit of a net is unique provided that the net is convergent.

<u>Theorem 6.2.</u> Let $S \subseteq X$, X be a (Hausdorff) topological space. Then
$$\overline{S} = \{ x \in X \mid \exists \{x_\mu\}_{\mu \in M} \underset{net}{\subseteq} S \text{ s.t. } x_\mu \to x \}.$$

<u>Sketch of the proof.</u> The less trivial part is the inclusion "$\subseteq$". Then for $\forall x \in \overline{S}$ the system $M_x := \{ U \mid U \text{ is a neihgbourhood of } x \}$ is a directed set and $U \cap S \neq \emptyset$ $\forall U \in M_x$. Then $x_u \to x$ where $x_u \in U \cap S$. $\square$

3. Operator topologies

We define these topologies in terms of convergence:

A uniformly convergent net:
$$A_\eta \rightrightarrows A \iff \| A_\eta - A \| \to 0$$
A strongly convergent net:
$$A_\eta \xrightarrow[s]{} A \iff \forall v \in \mathcal{H}: \ A_\eta v \to Av \quad (\text{i.e. } \| A_\eta v - Av \| \to 0)$$
A weakly convergent net:
$$A_\eta \xrightarrow[w]{} A \iff \forall u, v \in \mathcal{H}: \ \langle u, A_\eta v \rangle \to \langle u, Av \rangle$$

This a general situation in locally convex topological vector spaces W: there is a family $\{ p_j : W \to [0, \infty) \}_{j \in J}$ of seminorms and a net $\{w_\eta\} \subseteq W$ converges to $w \in W$ iff $\forall j \in J$ $p_j(w_\eta - w) \to 0$.
For the uniform operator topology we have one only norm, i.e. it is Banach space; for the strong operator topology the system of seminorms is $\{ A \mapsto \|Av\| \}_{v \in \mathcal{H}}$ and for the weak topology: $\{ A \mapsto |\langle u, Av \rangle| \}_{u, v \in \mathcal{H}}$.

The strong topology on $\mathcal{B}(\mathcal{H})$ is generated by a basis of neighbourhoods: $A + \{ C \in \mathcal{B}(\mathcal{H}) \mid \| C v_k \| < \varepsilon_k, \ k = 1, \dots, n \}$
for $\forall n = 1, 2, \dots$, $v_1, \dots, v_n \in \mathcal{H}$, $\varepsilon_1, \dots, \varepsilon_n > 0$.

The weak topology on $\mathcal{B}(\mathcal{H})$ is generated by a basis of neighbourhoods: $A + \{ C \in \mathcal{B}(\mathcal{H}) \mid |\langle u_k, C v_k \rangle| < \varepsilon_k, \ k = 1, \dots, n \}$
for $\forall n = 1, 2, \dots$, $u_1, \dots, u_n, v_1, \dots, v_n \in \mathcal{H}$, $\varepsilon_1, \dots, \varepsilon_n > 0$.

<u>Proposition. 6.3</u> (a) $A_\eta \rightrightarrows A \Rightarrow A_\eta \xrightarrow[s]{} A$; $A_\eta \xrightarrow[s]{} A \Rightarrow A_\eta \xrightarrow[w]{} A$.
For $S \subseteq \mathcal{B}(\mathcal{H})$ if $\overline{S}$, $\overline{S}^s$, $\overline{S}^w$ stand for the closures in the uniform, strong and weak topologies, respectively, then: $\overline{S} \subseteq \overline{S}^s \subseteq \overline{S}^w$

(b) If $A_\eta \xrightarrow[s]{} A$, $B \in \mathcal{B}(\mathcal{H})$ then

$$A_\eta B \xrightarrow[s]{} AB, \quad BA_\eta \xrightarrow[s]{} BA, \quad [A_\eta, B] \xrightarrow[s]{} [A, B]$$

(c) If $A_\eta \xrightarrow[w]{} A$, $B \in \mathcal{B}(\mathcal{H})$ then

$$A_\eta B \xrightarrow[w]{} AB, \quad BA_\eta \xrightarrow[w]{} BA, \quad [A_\eta, B] \xrightarrow[w]{} [A, B]$$

<u>Proof.</u> (a) Since: $\|A_\eta - A\| \geq \|v\|^{-1} \|A_\eta v - Av\|$,

$$\|A_\eta v - Av\| \geq \|u\|^{-1} |\langle u, A_\eta v\rangle - \langle u, Av\rangle|.$$

Then the inclusions $\overline{S} \subseteq \overline{S}^s \subseteq \overline{S}^w$ follow from Proposition 6.2.

(b) $A_\eta B v = A_\eta (Bv) \to ABv$, $\|BA_\eta v - BAv\| \leq \|B\| \, \|A_\eta v - Av\| \to 0$.

(c) $\langle u, A_\eta B v\rangle \to \langle u, A(Bv)\rangle$, $\langle u, BA_\eta v\rangle = \langle B^* u, A_\eta v\rangle \to \langle B^* u, Av\rangle$
$= \langle u, BAv\rangle$. □

<u>Def.</u> σ-strong $A_\eta \xrightarrow[\sigma s]{} A$) and σ-weak $(A_\eta \xrightarrow[\sigma w]{} A)$ topologies:

$$A_\eta \xrightarrow[\sigma s]{} A \iff \sum_{k=1}^{\infty} \|A_\eta v_k\|^2 \to \sum_{k=1}^{\infty} \|Av_k\|^2$$

for $\forall \{v_k\}_{k=1}^{\infty} \subseteq \mathcal{H}$ s.t. $\sum_{k=1}^{\infty} \|v_k\|^2 < \infty$

$$A_\eta \xrightarrow[\sigma w]{} A \iff \sum_{k=1}^{\infty} \langle u_k, A_\eta v_k\rangle \to \sum_{k=1}^{\infty} \langle u_k, Av_k\rangle$$

for $\forall \{u_k\}_{k=1}^{\infty}, \{v_k\}_{k=1}^{\infty} \subseteq \mathcal{H}$ s.t. $\sum_{k=1}^{\infty} \|u_k\|^2 < \infty$, $\sum_{k=1}^{\infty} \|v_k\|^2 < \infty$

The corresponding systems of seminorms are:

$A \mapsto \sum_{k=1}^{\infty} \|Av_k\|^2$ and $A \mapsto \sum_{k=1}^{\infty} \langle u_k, Av_k\rangle$, respectively.

Note that the above systems of seminorms are directed (recall, a directed system $\{p_j\}_{j\in J}$ of seminorms is s.t. $\forall j_1, j_2 \in J \; \exists j \in J : p_j \leq p_{j_1}, p_j \leq p_{j_2}$).
Hence, the bases of neighbourhoods of $A \in \mathcal{B}(\mathcal{H})$ are:

in σ-strong topology $\quad A + \{C \in \mathcal{B}(\mathcal{H}) \mid \sum_{k=1}^{\infty} \|Av_k\|^2 < \varepsilon\}$

in σ-weak topology $\quad A + \{C \in \mathcal{B}(\mathcal{H}) \mid \sum_{k=1}^{\infty} |\langle u_k, Av_k\rangle| < \varepsilon\}$

for $\forall \{u_k\}_{k=1}^{\infty}, \{v_k\}_{k=1}^{\infty} \subseteq \mathcal{H}$ s.t. $\sum_{k=1}^{\infty} \|u_k\|^2 < \infty$, $\sum_{k=1}^{\infty} \|v_k\|^2 < \infty$

Remark. Note that from every system $\{p_j\}_{j\in J}$ of seminorms we can obtain a directed system by taking all possible finite sums $p_{j_1}+\dots+p_{j_n}$. The resulting system $\{p_{j_1}+\dots+p_{j_n}\}_{n\in\mathbb{N},\, j_1,\dots,j_n\in J}$ induces the same topology.

The the passages strong $\to$ σ-strong and weak $\to$ σ-weak correspond to passing to suitable infinite sums $\sum_{k=1}^{\infty} p_{j_k}$.

The following construction relates the strong with σ-strong and weak with σ-weak topologies:

Let $\widehat{\mathcal{H}} := \bigoplus_{k=1}^{\infty} \mathcal{H}$ – the Hilbert space of all sequences $\underline{v} := \{v_k\}_{k=1}^{\infty} \subseteq \mathcal{H}$ s.t. $\|\underline{v}\|^2 := \sum_{k=1}^{\infty} \|v_k\|^2 < \infty$. The scalar product is $\langle \underline{u}, \underline{v}\rangle := \sum_{k=1}^{\infty} \langle u_k, v_k\rangle$.

Check as an exercise that $\widehat{\mathcal{H}}$ is indeed a Hilbert space.

Define: $\pi : \mathcal{B}(\mathcal{H}) \to \mathcal{B}(\widehat{\mathcal{H}})$ by $\pi(A)\{v_k\}_{k=1}^{\infty} := \{Av_k\}_{k=1}^{\infty}$.

This is clearly an injective morphism of unital $*$-algebras.

Lemma 6.4. (a) For a net $\{A_\eta\} \subseteq \mathcal{B}(\mathcal{H})$:

$$A_\eta \xrightarrow[\sigma s]{} A \iff \pi(A_\eta) \xrightarrow[s]{} \pi(A) \; ; \; A_\eta \xrightarrow[\sigma w]{} A \iff \pi(A_\eta) \xrightarrow[w]{} \pi(A)$$

(b) $\forall S \subseteq \mathcal{B}(\mathcal{H})$: $\pi(S'') = \pi(S)''$

Proof. (a) is straightforward from definitions.

(b) Let $U_k : \mathcal{H} \to \widehat{\mathcal{H}}$ – the embedding on the k-th copy of $\mathcal{H}$ in $\widehat{\mathcal{H}}$.

Then $U_k^* : \widehat{\mathcal{H}} \to \mathcal{H}$ – the projection on the k-th copy.

1.) $\hat{C} \in \pi(\mathcal{B}(\mathcal{H})) \subseteq \mathcal{B}(\widehat{\mathcal{H}}) \iff [\hat{C}, U_k U_\ell^*] = 0 \;\; \forall k, \ell$

$\Rightarrow$) obvious – check! $\Leftarrow$) Since $U_\ell^* U_k = 0$ for $k \neq \ell$, $U_k^* U_k = 1_{\mathcal{H}}$ then

$$[\hat{C}, U_k U_\ell^*] = 0 \iff \hat{C} U_k U_\ell^* = U_k U_\ell^* \hat{C} \xRightarrow{U_k^* \,|\, U_\ell} U_k^* \hat{C} U_k = U_\ell^* \hat{C} U_\ell .$$

Set $C := U_k^* \hat{C} U_k$ (independent in k). $P_k := U_k U_k^*$ – orthogonal projections on $\widehat{\mathcal{H}}$ that form a partition of unity. Check then that $P_k(\hat{C} - \pi(C))P_\ell = 0 \;\forall k,\ell$. $\Rightarrow \hat{C} = \pi(C)$.

2) By 1) $U_k U_\ell^* \in \pi(S)'$. $\Rightarrow \pi(S)'' \subseteq \{U_k U_\ell^* | k,\ell\}'$. By 1) $\Rightarrow \pi(S)'' \subseteq \pi(\mathcal{B}(\mathcal{H}))$.

But $\pi : \mathcal{B}(\mathcal{H}) \cong \pi(\mathcal{B}(\mathcal{H}))$. $\Rightarrow \pi(S'') = \pi(S)''$ (why?). □

4. Bicommutant theorem of von Neumann

Theorem Let $\mathcal{A} \subseteq \mathcal{B}(\mathcal{H})$ be a nondegenerate $*$-subalgebra. Then:

$$\mathcal{A}'' = \overline{\mathcal{A}}^{w} = \overline{\mathcal{A}}^{s} = \overline{\mathcal{A}}^{\sigma w} = \overline{\mathcal{A}}^{\sigma s}$$

Proof. By Proposition 6.3 and Theorem 6.2 : $\mathcal{A}'' \supseteq \overline{\mathcal{A}}^{w} \supseteq \overline{\mathcal{A}}^{s}$

$$\begin{array}{ccc} & \cup\!\!\shortmid & \cup\!\!\shortmid \\ & \overline{\mathcal{A}}^{\sigma w} \supseteq & \overline{\mathcal{A}}^{\sigma s} \end{array}$$

. It remains to prove the inclusion $\overline{\mathcal{A}}^{\sigma s} \supseteq \mathcal{A}''$. We split this in several steps:

1) $\forall v \in \mathcal{H}: v \in \overline{\mathcal{A} \cdot v}$.

Indeed: let P be the orthogonal projection on $\overline{\mathcal{A}\cdot v}$ and let $v' = Pv$, v' $(1-P)v \in \overline{\mathcal{A}\cdot v}$. Since $\overline{\mathcal{A}\cdot v}$ is $\mathcal{A}$-invariant then $P \in \mathcal{A}'$. Then $\forall A \in \mathcal{A}$: $Av'' = A(1-P)v = Av - PAv = 0$. Hence, $\forall u \in \mathcal{A}\cdot\mathcal{H}$: $u = Aw$ for $A \in \mathcal{A}, w \in \mathcal{H}$ and $\langle u, v'\rangle = \langle Aw, v'\rangle = \langle w, A^* v'\rangle = 0$. Since $\overline{\mathcal{A}\cdot\mathcal{H}} = \mathcal{H}$ ($\mathcal{A}$ is nondegenerate) then $v' = (1-P)v = 0$. (Recall also: $v' \in \{u \in \mathcal{H} \mid \mathcal{A}.u = 0\} = \mathcal{A}\mathcal{H}^{\perp} = 0$.)

2.) $\forall v \in \mathcal{H}$: $\overline{\mathcal{A}\cdot v} = \overline{\mathcal{A}''\cdot v}$.

Indeed: the inclusion "$\subseteq$" is obvious. Let $P :=$ the orthogonal projection on $\overline{\mathcal{A}\cdot v}$ and $P \in \mathcal{A}'$ as in 1.). Then $P \in (\mathcal{A}'')' = \mathcal{A}'$. By Lemma 6.1: $\mathcal{A}''(\overline{\mathcal{A}\cdot v}) \subseteq \overline{\mathcal{A}\cdot v}$. But $v \in \overline{\mathcal{A}\cdot v}$. $\Rightarrow$ $\overline{\mathcal{A}''\cdot v} \subseteq \overline{\mathcal{A}\cdot v}$.

3) Let $C \in \mathcal{A}''$. By Lemma 6.4 $\pi(\mathcal{A}'') = \pi(\mathcal{A})''$ $\left(\pi(\overline{\mathcal{A}}^{\sigma s}) = \overline{\pi(\mathcal{A})}^{s}\right)$. By 2): $\overline{\pi(\mathcal{A})\cdot \underline{v}} = \overline{\pi(\mathcal{A})''\cdot \underline{v}}$ $(\forall \underline{v} \in \widehat{\mathcal{H}})$. Then $\forall \varepsilon > 0$ $\forall \underline{v} \in \widehat{\mathcal{H}}$: $\exists A \in \mathcal{A}$ s.t. $\|\pi(C)\cdot\underline{v} - \pi(A)\cdot\underline{v}\| < \varepsilon$. $\Rightarrow$ $\forall$ basic neighbourhood of C in the σ-strong topology intersects $\mathcal{A}$. $\Rightarrow$ $C \in \overline{\mathcal{A}}^{\sigma s}$, i.e. $\mathcal{A}'' \subseteq \overline{\mathcal{A}}^{\sigma s}$. $\square$

5. The operator order in von Newmann algebras.

5a) Positive elements in C^*-algebras

We defined in Lect. 2 the positive element in a C^*-algebra $\mathcal{A}$ by:

$$a \geq 0 \iff a = c_1^* c_1 + \cdots + c_n^* c_n \quad \text{for some } c_1, \ldots, c_n$$

Theorem In a (unital) C^*-algebra $\mathcal{A}$:

$$a \geq 0 \iff \exists c \in \mathcal{A},\ c^* = c,\ c^2 = a.$$

Proof Enough to prove that $\forall a, b \in \mathcal{A}$, $\exists c \in \mathcal{A}$, $c^* = c$ and: $a^*a + b^*b = c^2$. $\forall \varepsilon > 0$, by Gelfand's theorem:

$\exists (\varepsilon.1 + a^*a)^{\pm 1/2} \in \mathcal{A}$, hermitian.

$\Rightarrow$

$$\varepsilon.1 + a^*a + b^*b = (\varepsilon.1 + a^*a)^{-1/2} \overbrace{\left(1 + (\varepsilon.1 + a^*a)^{1/2} b^*b (\varepsilon.1 + a^*a)^{1/2}\right)}^{\text{again by Gelfand's thm. } = d_\varepsilon^2 \text{ for } d_\varepsilon^* = d_\varepsilon} (\varepsilon 1 + a^*a)^{-1/2}$$

$\Rightarrow$ $\varepsilon.1 + a^*a + b^*b = \tilde{c}_\varepsilon^* \tilde{c}_\varepsilon$ for $c_\varepsilon = d_\varepsilon (\varepsilon.1 + a^*a)^{-1/2}$.

By the Gelfand theorem we find c_ε – hermitian s.t. $\tilde{c}_\varepsilon^* \tilde{c}_\varepsilon = c_\varepsilon^2$.

Thus $c_\varepsilon^2 = \varepsilon.1 + a^*a + b^*b$, but $c_{\varepsilon_1}^2 - c_{\varepsilon_2}^2 = (\varepsilon_1 - \varepsilon_2).1$

$\Rightarrow$ $\|c_{\varepsilon_1} - c_{\varepsilon_2}\| = \left\|\left((\varepsilon_1 - \varepsilon_2).1 + c_{\varepsilon_2}^2\right)^{1/2} - c_{\varepsilon_2}\right\| \leq \sqrt{\varepsilon_1 - \varepsilon_2} \xrightarrow[\varepsilon_1, \varepsilon_2 \to 0]{} 0$

(since $\sup_{x \in [0,\infty)} \left(\sqrt{\varepsilon + x^2} - x\right) = \sqrt{\frac{\varepsilon}{3}} < \sqrt{\varepsilon}$). Thus, $\exists c = \lim_{\varepsilon \to 0} c_\varepsilon$ and

$c = c^*$, $c^2 = \lim_{\varepsilon \to 0} c_\varepsilon^2 = a^*a + b^*b$. $\square$

5b) Positive operators.

In a Hilbert space $\mathcal{H}$ we have an alternative notion of positivity:

$A \in \mathcal{B}(\mathcal{H})$ is a positive operator $\Leftrightarrow$ $\forall v \in \mathcal{H}$: $\langle v, Av \rangle \geq 0$.

In particular, $A^* = A$ (why?). We shall prove below that $\exists C \in \mathcal{B}(\mathcal{H})$, $C^* = C$, $C^2 = A$.

Def. Introduce an order in $\mathcal{B}(\mathcal{H})$: $A \geq B$ iff $A - B$ is a positive operator, i.e. $\forall v \in \mathcal{H}$: $\langle v, Av \rangle \geq \langle v, Bv \rangle$. If B is also positive then it follows $\|A\| \geq \|B\|$ since $\forall v \in \mathcal{H}$ $|\langle v, Bv \rangle| \leq \langle v, Av \rangle \leq \|A\| \|v\|^2$.

Theorem 6.5 Let $\{A_\eta\}_{\eta \in M} \subseteq \mathcal{B}(\mathcal{H})$ be a net of postive operators, which is downward directed, i.e. $\eta \geq \eta' \Rightarrow A_\eta \leq A_{\eta'}$. Then
(насочена надолу)
$\exists A \in \mathcal{B}(\mathcal{H})$ s.t.: $\forall v \in \mathcal{H}$: $\langle v, Av \rangle = \inf_{\eta \in M} \langle v, A_\eta v \rangle$. Moreover, $A_\eta \xrightarrow{s} A$.

In particular, $\forall$ decrearing sequence $A_1 \geq A_2 \geq \cdots$ of positive operators has an infimum.

Proof. Let $A_\eta \geq A_{\eta'}$. Then $\langle u, (A_\eta - A_{\eta'}) v \rangle$ is a positive Hermitian form on $\mathcal{H}$. By the Cauchy-Schwarz inequality: $|\langle u, (A_\eta - A_{\eta'}) v \rangle|^2$
$\leq \langle u, (A_\eta - A_{\eta'}) u \rangle \langle v, (A_\eta - A_{\eta'}) v \rangle \leq \|A_\eta - A_{\eta'}\| \|u\|^2 \langle v, (A_\eta - A_{\eta'}) v \rangle$.

Setting $u = (A_\eta - A_{\eta'})v$: $\|(A_\eta - A_{\eta'})v\|^2 \leq \|A_\eta - A_{\eta'}\| \langle v, (A_\eta - A_{\eta'})v\rangle$. Also then $A_\eta \geq A_{\eta'} \Rightarrow \|A_\eta\| \geq \|A_{\eta'}\|$. Then: $\exists \lim_\eta \|A_\eta\| = \inf_\eta \|A_\eta\|$. Since $\langle v, A_\eta v\rangle$ is downward directed sequence $\Rightarrow \exists \lim_\eta \langle v, A_\eta v\rangle = \inf_\eta \langle v, A_\eta v\rangle$.
$\Rightarrow \forall v \in \mathcal{H}\ \exists \lim_\eta A_\eta v =: Av$. Since $\|Av\| = \lim_\eta \|A_\eta v\| \leq \lim_\eta \|A_\eta\| \|v\|$
$= (\inf_\eta \|A_\eta\|)\|v\|$, then $A \in \mathcal{B}(\mathcal{H})$. Thus, $A_\eta \xrightarrow{s} A$. $\Rightarrow A_\eta \xrightarrow{w} A$,
i.e. $\langle v, Av\rangle = \lim_\eta \langle v, A_\eta v\rangle = \inf_\eta \langle v, A_\eta v\rangle$. $\square$

<u>Corollary 6.6.</u> Every net $\{A_\eta\} \subseteq \mathcal{B}(\mathcal{H})$ of Hermitian operators, which is downward/upward directed and bounded from below/above is strongly convergent in $\mathcal{B}(\mathcal{H})$, i.e. $A_\eta \xrightarrow{s} A$, and
$$\inf_\eta \langle v, A_\eta v\rangle = \langle v, Av\rangle \quad \left(\text{resp., } \sup_\eta \langle v, A_\eta v\rangle = \langle v, Av\rangle\right).$$

<u>Theorem 6.7</u> Let $A \in \mathcal{B}(\mathcal{H})$ and A be a positive operator. Then $\exists C \in \mathcal{B}(\mathcal{H})$ s.t. $C^* = C$, $C^2 = A$. Hence, A is a positive element of the C^*-algebra $\mathcal{B}(\mathcal{H})$.
(Note: the converse is trivial: $\forall$ positive element of the C^*-algebra $\mathcal{B}(\mathcal{H})$ is a positive operator.)

<u>Proof.</u> Note first that $A \leq \|A\| \cdot \mathbb{1}$. Then replacing $A \mapsto \|A\|^{-1}A$ we can assume that $0 \leq A \leq \mathbb{1}$. Consider the operator equation $X = \frac{1}{2}(B + X^2)$ with $B = 1 - A$ - positive operator. Its solution X satisfies then $(1-X)^2 = A$, so if $X = X^*$ we can set $C = 1 - X$.
Let us define inductively: $X_{n+1} = \frac{1}{2}(B + X_n^2)$, $X_0 = 0$.
By induction in n: $X_n = p_n(B)$ where $p_n(t)$ is a polynomial with nonnegative cofficients. Since $X_{n+1} - X_n = \frac{1}{2}(X_n^2 - X_{n-1}^2) =$
$= \frac{1}{2}(X_n + X_{n-1})(X_n - X_{n-1})$ then $X_{n+1} - X_n = q_n(B)$, where $q_n(t)$ is again a polynomial with nonnegative coefficients. But B^n is a positive operator $\forall n = 0, 1, 2, \ldots$ (why?). By induction in $n = 0, 1, \ldots$ $\|X_n\| \leq 1$ (and hence, $1 - X_n$ are positive operators): indeed, $\|X_0\| = 0$, $\|B\| = \|1 - A\| \leq 1$
$\|X_{n+1}\| \leq \frac{1}{2}(\|B\| + \|X_n\|^2)$ then $\|X_n\| \leq$ smallest root of $t = \frac{1}{2}(\|B\| + t^2)$, which is $t' = 1 - \sqrt{1 - \|B\|} \leq 1$ since if $t_0 = 0$, $t_{n+1} = \frac{1}{2}(\|B\| + t_n^2)$ then $t_n \uparrow t'$ and $\|X_n\| \leq t_n$.

Thus, $\{1-X_n\}_{n=0}^{\infty}$ is a sequence of positive operators, which is downward directed. $\Rightarrow 1-X_n \xrightarrow[s]{} C$ and $C^2 = A$ (why?). Since $1-X_n \xrightarrow[w]{} C$ and $X_n^* = X_n$ then $C^* = C$ (why?). $\square$

<u>Corollary 6.8</u> In $\forall$ von Neumann algebra $\mathfrak{M} \subseteq \mathcal{B}(\mathcal{H})$, and $\forall$ net $\{A_\lambda\} \subseteq \mathfrak{M}$, which is bounded from above and directed upwards, has a supremum $\sup_\lambda A_\lambda \in \mathfrak{M}$.

5c.) *Spectral decomposition in commutative von Neumann*

Let $S \subseteq \mathcal{B}(\mathcal{H})$ be a $*$-invariant subset of mutually commuting operators. If $\mathfrak{A}$ is the C^*-subalgebra generated by S then $\mathfrak{A}$ is commutative and $S' = \mathfrak{A}'$. Then $S'' = \mathfrak{A}'' = \overline{\mathfrak{A}}^s$ and hence, S'' is a commutative (why?) von Neumann algebra.

In particular, if $A \in \mathcal{B}(\mathcal{H})$, $A^* = A$ and take $S = \{A\}$, then $\mathfrak{A} = \overline{\mathbb{C}[A]}$ and $S'' = \mathfrak{A}'' = \overline{\mathbb{C}[A]}^s$ - this is the commutative von Neumann algebra generated by A. We shall show now that it contains all the spectral projections of A.

Recall, by the Gelfand theorem (Prop. 4.13): $\overline{\mathbb{C}[A]} \cong C(\sigma_{\mathfrak{A}}(A))$. In particular, this provides us with a notion of a continuous function of A: $f(A) \longleftrightarrow f(t) \in C(\sigma_{\mathfrak{A}}(A))$ under the above isomorphism. The spectral projections P_I for an interval $I \subseteq \mathbb{R}$ would correspond to $\chi_I(A)$ where $\chi_I(t)$ is the characteristic function of I: $\chi_I(t) = 1$ for $t \in I$ and $\chi_I(t) = 0$ for $t \in \mathbb{R} \setminus I$. But $\chi_I(t) \notin C(\sigma_{\mathfrak{A}}(A))$ in general so that $P_I \notin \overline{\mathbb{C}[A]}$. But $P_I \in \overline{\mathbb{C}[A]}^s = \overline{\mathbb{C}[A]}''$ since χ_I can be obtained as a supremum of a bouded increasing sequence of continuous functions $f_1 \leq f_2 \leq \cdots \leq \chi_I$. More generally, every Borel function $F(t)$ (i.e. a measurable function w.r.t. to Borel σ-algebra on $\mathbb{R}$) can be approximated by infimums or supremums of smooth functions. And, thus $F(A)$ can be defined for $F(t)$.

<u>Remark</u>. This is the way in which one can generally prove that $\forall$ commutative von Neumann algebra is $\cong L^\infty(X)$ - the algebra of all essentially bounded functions on a space with measure.
For the basic measure theory: Rudin, Real and Complex Analysis Chap. 1 / Рудин, Реален и комплексен анализ, Глава 1.

6. Further characteristic properties of von Veumann algebras

6a) A C^*-algebra $\mathcal{A}$ is isomorphic to a von Neumann algebra iff it is contained in the uniform closure of the algebra generated by all its projections (i.e. all $h \in \mathcal{A}$ s.t. $h^* = h$ and $h^2 = h$).

Note that for the von Neumann algebra $C[A]'' \subseteq \mathcal{B}(\mathcal{H})$ for $A = A^*$ we have $C[A]'' = \overline{\operatorname{Span}\{P_I \mid P_I \text{ - spectral projection}\}}$ since $\forall$ measurable function can be uniformly approximated by simple functions - finite linear combinations of characteristic functions.

6b) Again: $\mathcal{Q}$ - unital C^*-algebra is $\cong$ von Numann algebra $\Longleftrightarrow$ $\mathcal{Q} \cong$ the dual space of a Banach space.
W^*-algebra $\overset{\text{def}}{\Longleftrightarrow}$ unital C^*-algebra $\cong$ the dual space of a Banach space.

The latter space is called predual space of $\mathcal{Q}$ and is denoted by $\mathcal{Q}_*$.

6c) Predual of a Banach space X: this is a Banach space Y together with an isomeric isomorphism $j : X \cong Y^*$

Note that then there is a nondegenerate pairing:
$X \times Y \to \mathbb{C} : (x, y) \mapsto \lambda(x, y) \equiv j(x)[y]$, which is nondegenerate $\forall x \exists y : \lambda(x,y) \neq 0$ & $\forall y \exists x \ \lambda(x,y) \neq 0$.
In particular, $y \mapsto \lambda(\cdot, y)$ gives an injection: $Y \hookrightarrow X^*$

Thus, $\forall$ predual of X (if there is any) has a canonical embeding in X

6d) Every von Neumann algebra $\mathcal{A}$ has a unique predual $\mathcal{A}_*$ in $\mathcal{A}^*$. The linear functionals $\varphi \in \mathcal{A}_* \subseteq \mathcal{A}^*$ are called normal linear functionals.

φ is normal $\Leftrightarrow$ φ is σ-strongly continuous

$\Leftrightarrow$ φ is σ-weak continuous $\Leftrightarrow$ $\exists\, C \in \mathcal{B}(\mathcal{H})$ s.t.

$$tr\left((C^*C)^{1/2}\right) < \infty$$

$$\varphi(A) = tr(CA) \quad \forall A \in \mathcal{A}$$

6e) $\mathcal{B}(\mathcal{H})_* = \{ C \in \mathcal{B}(\mathcal{H}) \mid tr\left((C^*C)^{1/2}\right) < \infty \}$

i.e. C is trace class

see below

Old notes from 2010:

Нека $\mathcal{H}$ е Хилбертово пр-во,

$\mathcal{B}(\mathcal{H}) := \{ A : \mathcal{H} \to \mathcal{H} \ : \ A$ - линеен, ограничен оператор $\}$.

Както знаем от лекция 1 $\mathcal{B}(\mathcal{H})$ е C^*-алгебра.

$\mathcal{B}(\mathcal{H})$ е и първият пример за алгебра на Фон Нойман, които ще определим по-късно.

В/у $\mathcal{B}(\mathcal{H})$ имаме специален вид линейни непрекъснати функционали - матричните елементи:

$$\lambda_{\varphi,\psi}(A) := \langle \varphi, A\psi \rangle \ , \text{ за } \varphi, \psi \in \mathcal{H}.$$

Така $\mathrm{Span}_{\mathbb{C}} \{ \lambda_{\varphi,\psi} : \varphi, \psi \in \mathcal{H} \} \subseteq \mathcal{B}(\mathcal{H})^*$

комплексна линейна обвивка (крайни комплексни линейни комбинации)

<u>Забележка:</u> $\| \lambda_{\varphi,\psi} \| = \|\varphi\| \, \|\psi\|$. В едната посока:

$$\| \lambda_{\varphi,\psi} \| \underset{\mathrm{def}}{=} \sup \{ | \lambda_{\varphi,\psi}(A) | : \|A\| = 1 \} =$$

$= \sup \{ |\langle \varphi, A\psi \rangle| : \|A\| = 1 \} \leq \|\varphi\| \, \|\psi\|$, а в обратна посока да се положи $A := \langle \psi, \cdot \rangle \varphi$ (упражнение).

Първата цел ще бъде да опишем $\overline{\mathrm{Span}_{\mathbb{C}} \{ \lambda_{\varphi,\psi} : \varphi, \psi \in \mathcal{H} \}}$ (затварянето е в Банаховото пространство $\mathcal{B}(\mathcal{H})^*$).

По-конкретно ще докажем, че

$$\overline{\mathrm{Span}_{\mathbb{C}} \{ \lambda_{\varphi,\psi} : \varphi, \psi \in \mathcal{H} \}} \cong \text{ Пространството от оператори в } \mathcal{H} \text{ от } tr \text{ (trace) - клас.}$$

След това ще покажем, че това пространство е преддуално на $\mathcal{B}(\mathcal{H})$.

Нека въведем за целта операторите $O_{\varphi,\psi} := \langle \varphi, \cdot \rangle \psi$, т.е.

$$O_{\varphi,\psi} : \mathcal{H} \to \mathcal{H} \quad : \quad O_{\varphi,\psi}(\theta) := \langle \varphi, \theta \rangle \psi \,, \quad \varphi, \psi, \theta \in \mathcal{H}.$$

Във физиката означават : $O_{\varphi,\psi} \equiv |\psi\rangle\langle\varphi|$. Тогава:

$$\lambda_{\varphi,\psi}(A) = \mathrm{Tr}\left(O_{\varphi,\psi} A\right) \underset{def}{=} \sum_k \langle e_k, O_{\varphi,\psi} A e_k \rangle \,,$$

където $\{e_k\}_k$ е ортонормиран базис в $\mathcal{H}$ (упражнение).

Операторите $O_{\varphi,\psi}$ са специален клас линейни оператори в $\mathcal{H}$ -оператори от ранг 1. Те не образуват линейно пространство. Тяхната алгебрична линейна обвивка

$$\{ \alpha_1 O_{\varphi_1,\psi_1} + \dots + \alpha_n O_{\varphi_n,\psi_n} \}$$

се нарича пр-во на оператори от краен ранг.

<u>Лема 1</u>: Всеки оператор K от краен ранг може да се представи във вида:

$$K = \sum_{j=1}^{r} \varkappa_j \, O_{e_j, f_j}$$

където $\{e_1, \dots, e_r\}$ и $\{f_1, \dots, f_r\}$ са ортонормирани системи от вектори в $\mathcal{H}$ и $\varkappa_1, \dots, \varkappa_r > 0$. При това:

$$K^* K = \sum_{j=1}^{r} \varkappa_j^2 \, O_{e_j, e_j} \,, \quad \text{т.е.}$$

$(K^*K)^{1/2}$ е оператор подобен на $\mathrm{diag}(\varkappa_1, \dots, \varkappa_r)$, т.е.
Този оператор се бележи с $|K| := (K^*K)^{1/2}$.

Упътване към доказателството:

Нека $K = \sum_{j=1}^{n} \alpha_j O_{\varphi_j, \psi_j}$

Да означим $\mathcal{H}_1 = \mathrm{Span}\{\psi_1, \dots, \psi_n\}$, $\mathcal{H}_2 = \mathrm{Span}\{\varphi_1, \dots, \varphi_n\}$

Така: $\mathcal{H}_1, \mathcal{H}_2 \subseteq \mathcal{H}$ и $K : \mathcal{H}_1 \to \mathcal{H}_2$, $K\big|_{\mathcal{H}_1^\perp} = 0$.

Тогава $K^* : \mathcal{H}_2 \to \mathcal{H}_1$ и $K^*K : \mathcal{H}_1 \to \mathcal{H}_1$ – положителен ермитов оператор в крайно-мерно хилбертово пространство.

Диагонализираме K^*K: $e_1, \dots, e_s$ – ортонормиран базис в $\mathcal{H}_1$ такъв, че $(K^*K) e_j = \varkappa_j^2 e_j$, където $\varkappa_1 > 0, \dots, \varkappa_r > 0$, $\varkappa_{r+1} = \dots = \varkappa_s = 0$.

Тогава $f_1, \dots, f_s$ избираме пропорционални на $Ke_1, \dots, Ke_s$.

(детайлите и проверките – упражнение). $\square$

<u>Лема 2</u> Нека K е оператор от краен ранг и

$$\lambda_K(A) := \mathrm{Tr}(KA) \text{ - дефинира } \lambda_K \in \mathcal{B}(\mathcal{H})^*.$$

Тогава $\|\lambda_K\| = \mathrm{Tr}|K| \equiv \sum_{j=1}^{r} \varkappa_j$, където

$$K = \sum_{j=1}^{r} \varkappa_j O_{e_j, f_j} \text{ – както в Лема 1.}$$

<u>Д-во:</u> $$\|\lambda_K\| \underset{def}{=} \sup\{|\mathrm{Tr}(KA)| : \|A\| = 1\} \leq$$

$$\leq \sup\Big\{\sum_{j=1}^{r} \varkappa_j \underbrace{|\langle e_j, Af_j\rangle|}_{\leq \|A\| \|e_j\| \|f_j\| = \|A\|} : \|A\| = 1\Big\} \leq \sum_{j=1}^{r} \varkappa_j$$

За обратното неравенство $\mathrm{Tr}(KA) = \sum_{j=1}^{r} \varkappa_j$, за $A := \sum_{j=1}^{r} O_{f_j, e_j}$ като $\|A\| = 1$ (проверете!). $\square$

И така, от Лема 2 следва, че

$$\overline{\operatorname{Span}_{\mathbb{C}}\{\lambda_{\varphi,\psi} : \varphi,\psi\in\mathcal{H}\}} \cong$$ Пространството от оператори в $\mathcal{H}$ от tr (trace) - клас.

При това един оператор K в $\mathcal{H}$ е по дефиниция от tr-клас, ако

$$Tr\left(K^*K\right)^{1/2} := \sum_{j=1}^{\infty} \langle e_j, K^*K e_j\rangle \equiv \sum_{j=1}^{\infty} \|Ke_j\|^2 < \infty$$

за поне един ортонормиран базис $\{e_j\}$ в $\mathcal{H}$.

<u>Забележка</u> K^*K е неотрицателен операторв $\mathcal{B}(\mathcal{H})$ - C^*-алгебра и $(K^*K)^{1/2}$ се определя съгласно теоремата на Гелфанд (защо?).

$\|K\|_1 := Tr\left(K^*K\right)^{1/2}$ е tr-нормата спрямо която

$\overline{\operatorname{Span}_{\mathbb{C}}\{\lambda_{\varphi,\psi} : \varphi,\psi\in\mathcal{H}\}}$ е Банахово подпространство в $\mathcal{B}(\mathcal{H})^*$.

Минаваме към втората задача - да покажем, че пр-вото от tr-клас оператори е преддуално на $\mathcal{B}(\mathcal{H})$, т.е. че $\mathcal{B}(\mathcal{H})$ е неговото дуално Банахово пр-во.

Пространството на оператори от tr-клас в $\mathcal{H}$ ще бележим $\mathcal{L}^1(\mathcal{H})$.

И така:

<u>Теорема.</u> $\mathcal{B}(\mathcal{H}) \cong \mathcal{L}^1(\mathcal{H})^*$.

<u>Д-во:</u> Вече знаем, че $\mathcal{L}^1(\mathcal{H}) \hookrightarrow \mathcal{B}(\mathcal{H})^*$ посредством съответствието: $K \mapsto \lambda_K := Tr(K\,\cdot)$

Забележка: Това показва в частност, че $\mathcal{L}^1(\mathcal{H})$ е двустранен идеал в $\mathcal{B}(\mathcal{H})$, т.е. че $\mathcal{L}^1(\mathcal{H}) . \mathcal{B}(\mathcal{H}) \subseteq \mathcal{L}^1(\mathcal{H})$ и $\mathcal{B}(\mathcal{H}) . \mathcal{L}^1(\mathcal{H}) \subseteq \mathcal{L}^1(\mathcal{H})$ (понеже $Tr(KA) = Tr(AK) < \infty$), но в това няма се задълбочаване тук.

Стратегията ни ще бъде: 1) да покажем, че непрекъснат линеен функционал в/у $\mathcal{L}^1(\mathcal{H})$ съответства изометрически на ограничена линейно-антилинейна 2-форма над $\mathcal{H}$.
"съответства изометрически" означава, че нормите се запазват при това съответствие.

2.) Остава да приложим базисния факт, че ограничените линейно-антилинейни 2-форми над $\mathcal{H}$ съответстват изометрично на ограничените оператори в $\mathcal{H}$.

Определение Ограничена линейно-антилинейна 2-форма над $\mathcal{H}$ $Q : \mathcal{H} \times \mathcal{H} \to \mathbb{C}$ е такава функция, която:

(а) е линейна по втория аргумент и антилинейна по първия:

$$Q(\varphi, \alpha\psi + \beta\theta) = \alpha Q(\varphi,\psi) + \beta Q(\varphi,\theta)$$

$$Q(\alpha\varphi + \beta\psi, \theta) = \bar{\alpha} Q(\varphi,\theta) + \bar{\beta} Q(\psi,\theta)$$

(b) Q е ограничена: $\|Q\| := \sup\{ |Q(\varphi,\psi)| : \|\varphi\| = \|\psi\| = 1 \} < \infty$.

Лема 3 Съответствието $A \mapsto Q_A$, $Q_A(\varphi,\psi) := \langle\varphi, A\psi\rangle$ е изометричен изоморфизъм м/у $\mathcal{B}(\mathcal{H})$ и Банаховото пр-во на ограничените линейно-антилинейни 2-форми над $\mathcal{H}$.

Д-во — упражнение (това, че $A \mapsto Q_A$ е "върху" следва от теоремата на Рис).

С това точка 2.) по-горе е изпълнена. Остава 1.)

Нека $\mu \in \mathcal{L}^1(\mathcal{H})^*$ – ограничен линеен ф-л. Тогава

$$Q_\mu(\varphi, \psi) := \mu(O_{\varphi,\psi}) \quad \text{за} \quad \varphi, \psi \in \mathcal{H},$$

дефинира линейно-антилинейна 2-форма над $\mathcal{H}$.

<u>Лема 4</u> Q_μ е ограничена и $\|Q_\mu\| = \|\mu\|$.

<u>Д-во:</u> Понеже $\|O_{\varphi,\psi}\| = \|\varphi\|\,\|\psi\|$, то

$$|Q_\mu(\varphi, \psi)| = |\mu(O_{\varphi,\psi})| \leq \|\mu\|\,\|O_{\varphi,\psi}\| = \|\mu\|\,\|\varphi\|\,\|\psi\|$$

$\Rightarrow$ $\|Q_\mu\| \leq \|\mu\|$. Обратното, ако $K = \sum_j \varkappa_j O_{e_j, f_j}$ е както в Лема 1, то: $|\mu(K)| = \left|\sum_j \varkappa_j Q_\mu(e_j, f_j)\right| \leq$

$$\leq \sum_j \varkappa_j |Q_\mu(e_j, f_j)| \leq \|Q_\mu\| \sum_j \varkappa_j = \|Q_\mu\|\,\|K\|_1 \text{, т.е.}$$

$|\mu(K)| \leq \|Q_\mu\|\,\|K\|_1$ за всеки оператор от краен ранг, но по построение пр-вото от тези оператори е гъсто в $\mathcal{L}^1(\mathcal{H})$.

$\Rightarrow$ по непрекъснатост: $|\mu(K)| \leq \|Q_\mu\|\,\|K\|_1, \ \forall K \in \mathcal{L}^1(\mathcal{H})$.

$\Rightarrow$ $\|\mu\| \leq \|Q_\mu\|$. $\square$

Това завършва д-вото на Теоремата. $\square$